

DEGREE AND VALUATION OF THE SCHUR ELEMENTS OF CYCLOTOMIC HECKE ALGEBRAS

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ABSTRACT. Following the generalization of the notion of families of characters, defined by Lusztig for Weyl groups, to the case of complex reflection groups, thanks to the definition given by Rouquier, we show that the degree and the valuation of the Schur elements (functions A and a) remain constant on the “families” of the cyclotomic Hecke algebras of the exceptional complex reflection groups. The same result has already been obtained for the groups of the infinite series and for some special cases of exceptional groups.

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Introduction

The work of G. Lusztig on the irreducible characters of reductive groups over finite fields has displayed the important role of the “families of characters” of the Weyl groups concerned. More recent results of Gyoja [12] and Rouquier [22] have made possible the definition of a substitute for families of characters which can be applied to all complex reflection groups. Rouquier has shown that the families of characters of a Weyl group W are exactly the blocks of characters of the Iwahori-Hecke algebra of W over a suitable coefficient ring, the “Rouquier ring”. This definition generalizes without problem to all cyclotomic Hecke algebras of complex reflection groups.

Since the families of characters of the Weyl group play an essential role in the definition of the families of unipotent characters of the corresponding finite reductive group (cf. [14]), we can hope that the families of characters of the cyclotomic Hecke algebras play a key role in the organization of families of unipotent characters more generally. Moreover, the determination of these families is crucial for the program “Spets” (cf. [4]), whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious objects.

In the case of the Weyl groups and their usual Hecke algebra, the families of characters can be defined using the existence of Kazhdan-Lusztig bases. Lusztig attaches to every irreducible character two integers, denoted by a and A , and shows (cf. [15], 3.3 and 3.4) that they are constant on the families. In an analogue way, we can define integers a and A attached to every irreducible character of a cyclotomic Hecke algebra of a complex reflection

group. For the groups of the infinite series, it has been shown that a and A are constant on the Rouquier blocks (cf. [3], [6], [8]). Moreover, Malle and Rouquier have proved that a and A are constant on the Rouquier blocks of the “spetsial” cyclotomic Hecke algebra of the “spetsial” exceptional complex reflection groups (cf. [20], Thm. 5.1). The aim of this paper is the proof of the same result for all cyclotomic Hecke algebras of all exceptional irreducible complex reflection groups.

In [5], we show that the Rouquier blocks of a cyclotomic Hecke algebra of any complex reflection group W depend on some numerical data of the group, its “essential hyperplanes”. These hyperplanes are defined by the factorization of the Schur elements of the generic Hecke algebra \mathcal{H} associated to W . We can associate a partition of the set $\text{Irr}(W)$ of irreducible characters of W to every essential hyperplane H , which we call “Rouquier blocks associated with the hyperplane H ”. Following theorem 4.3 and proposition 4.5, these partitions generate the partition of $\text{Irr}(W)$ into Rouquier blocks. They have been determined for all exceptional irreducible complex reflection groups in [5]. We have stored these data in a computer file and created the GAP function *AllBlocks* which displays them. We have also created the function *RouquierBlocks* which calculates the Rouquier blocks of a given cyclotomic Hecke algebra.

Let ϕ be a cyclotomic specialization and \mathcal{H}_ϕ the corresponding cyclotomic Hecke algebra. For every irreducible character, we define a and A to be, respectively, the valuation and the degree of the corresponding Schur element in \mathcal{H}_ϕ . In order to show that a and A are constant on the Rouquier blocks, we introduce the notions of “generic valuation” and “generic degree” (definition 5.8). Then corollary 5.10 in combination with proposition 4.5 imply that it is enough to check whether they remain constant on the Rouquier blocks associated with each essential hyperplane.

We have created a GAP program which verifies that the generic valuation and the generic degree remain constant on the Rouquier blocks for the groups G_7 , G_{11} , G_{19} , G_{26} , G_{28} and G_{32} . We provide the algorithm in section 6.1. Then Clifford theory allows us to extend this result to the groups G_4, \dots, G_{22} and G_{25} . Finally, in section 6.2, we explain why, for the remaining exceptional irreducible complex reflection groups, it is enough to check whether the functions a and A remain constant on the Rouquier blocks of the “spetsial” cyclotomic Hecke algebra.

All computer algorithms presented in this article require the GAP package CHEVIE, where, together with Jean Michlel, we have programmed the generic Schur elements of the exceptional irreducible complex reflection groups in a factorized form (functions *SchurModels* and *SchurData*). Moreover, they require the GAP functions *AllBlocks* and *RouquierBlocks* contained in the file “RouquierBlocks.g”. All the above, along with a program implementing the algorithms, can be found on my webpage: <http://www.math.jussieu.fr/~chlouveraki>.

1 Generalities on blocks

Let \mathcal{O} be a Noetherian and integrally closed domain with field of fractions F . Let A be an \mathcal{O} -algebra free and finitely generated as an \mathcal{O} -module.

Definition 1.1 *The blocks of A are the central primitive idempotents of A .*

Let K be a finite Galois extension of F such that the algebra $KA := K \otimes_{\mathcal{O}} A$ is split semisimple. Then there exists a bijection between the set $\text{Irr}(KA)$ of irreducible characters of KA and the set $\text{Bl}(KA)$ of blocks of KA which sends every irreducible character χ to the central primitive idempotent e_{χ} .

Theorem 1.2

1. We have $1 = \sum_{\chi \in \text{Irr}(KA)} e_{\chi}$ and the set $\{e_{\chi}\}_{\chi \in \text{Irr}(KA)}$ is the set of all the blocks of the algebra KA .
2. There exists a unique partition $\text{Bl}(A)$ of $\text{Irr}(KA)$ such that
 - (a) For all $B \in \text{Bl}(A)$, the idempotent $e_B := \sum_{\chi \in B} e_{\chi}$ is a block of A .
 - (b) We have $1 = \sum_{B \in \text{Bl}(A)} e_B$ and for every central idempotent e of A , there exists a subset $\text{Bl}(A, e)$ of $\text{Bl}(A)$ such that

$$e = \sum_{B \in \text{Bl}(A, e)} e_B.$$

In particular the set $\{e_B\}_{B \in \text{Bl}(A)}$ is the set of all the blocks of A . If $\chi \in B$ for some $B \in \text{Bl}(A)$, we say that “ χ belongs to the block e_B ”.

Now let us suppose that there exists a symmetrizing form on A , i.e., a linear map $t : A \rightarrow \mathcal{O}$ such that

- $t(aa') = t(a'a)$ for all $a, a' \in A$,
- the map

$$\begin{aligned} \hat{t} : A &\rightarrow \text{Hom}_{\mathcal{O}}(A, \mathcal{O}) \\ a &\mapsto (x \mapsto t(ax)) \end{aligned}$$

is an isomorphism of A -modules- A .

Then we have the following result due to Geck (cf. [10]).

Proposition 1.3

1. We have

$$t = \sum_{\chi \in \text{Irr}(KA)} \frac{1}{s_\chi} \chi,$$

where s_χ is the Schur element associated to χ .

2. For all $\chi \in \text{Irr}(KA)$, the central primitive idempotent associated to χ is

$$e_\chi = \frac{\hat{t}^{-1}(\chi)}{s_\chi}.$$

2 Generic Hecke algebras

Let μ_∞ be the group of all the roots of unity in \mathbb{C} and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K . For every integer $d > 1$, we set $\zeta_d := \exp(2\pi i/d)$ and denote by μ_d the group of all the d -th roots of unity.

Let V be a K -vector space of finite dimension r . Let W be a finite subgroup of $\text{GL}(V)$ generated by (pseudo-)reflections acting irreducibly on V . Let us denote by \mathcal{A} the set of the reflecting hyperplanes of W . For every orbit \mathcal{C} of W on \mathcal{A} , we denote by $e_{\mathcal{C}}$ the common order of the subgroups W_H , where H is any element of \mathcal{C} and W_H the subgroup formed by id_V and all the reflections fixing the hyperplane H .

We choose a set of indeterminates $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \leq j \leq e_{\mathcal{C}}-1)}$ and we denote by $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates \mathbf{u} . If we denote by B the braid group associated to W , then we define the *generic Hecke algebra* \mathcal{H} of W to be the quotient of the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \dots (\mathbf{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where \mathcal{C} runs over the set \mathcal{A}/W and \mathbf{s} runs over the set of monodromy generators around the images in \mathcal{M}/W of the elements of the hyperplane orbit \mathcal{C} .

Example 2.1 Let $W := G_4 = \langle s, t \mid sts = tst, s^3 = t^3 = 1 \rangle$. Then s and t are conjugate in W and their reflecting hyperplanes belong to the same orbit in \mathcal{A}/W . The generic Hecke algebra of W can be presented as follows

$$\mathcal{H}(G_4) = \langle S, T \mid STS = TST, (S - u_0)(S - u_1)(S - u_2) = 0, \\ (T - u_0)(T - u_1)(T - u_2) = 0 \rangle.$$

From now on, we assume that the algebra \mathcal{H} is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ -module of rank $|W|$ and that there exists a symmetrizing form t on \mathcal{H} which satisfies certain conditions (cf., for example, [3], Hyp. 2.1). Note that the above assumptions have been verified for all but a finite number of irreducible complex reflection groups ([4], remarks before 1.17, § 2; [11]).

Then we have the following result by G.Malle ([18], 5.2).

Theorem 2.2 Let $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C} \in \mathcal{A}/W)(0 \leq j \leq e_{\mathcal{C}}-1)}$ be a set of $\sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}$ indeterminates such that, for every \mathcal{C}, j , we have

$$v_{\mathcal{C},j}^{|\mu(K)|} := \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}.$$

The field K is the field of definition of W and the element $\zeta_{e_{\mathcal{C}}}$ belongs to K . We have that the $K(\mathbf{v})$ -algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple.

Example 2.3 If $W = G_4$ and $K = \mathbb{Q}(\zeta_3)$, then, in the example 2.1, we replace u_0, u_1, u_2 by $v_0^6, \zeta_3 v_1^6, \zeta_3^2 v_2^6$. The algebra $\mathbb{Q}(\zeta_3, v_0, v_1, v_2)\mathcal{H}(G_4)$ is split semisimple.

By “Tits’ deformation theorem” (cf., for example, [4], 7.2), it follows that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi \mapsto \chi_{\mathbf{v}}$ from the set $\text{Irr}(K(\mathbf{v})\mathcal{H})$ of absolutely irreducible characters of $K(\mathbf{v})\mathcal{H}$ to the set $\text{Irr}(W)$ of absolutely irreducible characters of W .

The following result concerning the form of the Schur elements associated with the irreducible characters of $K(\mathbf{v})\mathcal{H}$ is proved in [5], Thm. 3.2.5, using case by case analysis (cf. [1], [2], [13], [16], [17], [19], [21], [23]) and Clifford theory.

Theorem 2.4 The Schur element $s_{\chi}(\mathbf{v})$ associated with the character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ is an element of $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ of the form

$$s_{\chi}(\mathbf{v}) = \xi_{\chi} N_{\chi} \prod_{i \in I_{\chi}} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- ξ_{χ} is an element of \mathbb{Z}_K ,
- $N_{\chi} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ is a monomial in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ such that $\sum_{j=0}^{e_{\mathcal{C}}-1} b_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,
- I_{χ} is an index set,
- $(\Psi_{\chi,i})_{i \in I_{\chi}}$ is a family of K -cyclotomic polynomials in one variable (i.e., minimal polynomials of the roots of unity over K),
- $(M_{\chi,i})_{i \in I_{\chi}}$ is a family of monomials in $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and if $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, then $\gcd(a_{\mathcal{C},j}) = 1$ and $\sum_{j=0}^{e_{\mathcal{C}}-1} a_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,
- $(n_{\chi,i})_{i \in I_{\chi}}$ is a family of positive integers.

This factorization is unique in $K[\mathbf{v}, \mathbf{v}^{-1}]$. Moreover, the monomials $(M_{\chi,i})_{i \in I_{\chi}}$ are unique up to inversion, whereas the coefficient ξ_{χ} is unique up to multiplication by a root of unity.

Example 2.5 Let us denote by θ the only irreducible character of degree 3 of G_4 . If v_0, v_1, v_2 are defined as in example 2.3, then we have

$$s_\theta(\mathbf{v}) = \prod_{i=0}^2 \Phi_4(v_i^2 v_{i+1}^{-1} v_{i+2}^{-1}) \Phi'_{12}(v_i^2 v_{i+1}^{-1} v_{i+2}^{-1}) \Phi''_{12}(v_i^2 v_{i+1}^{-1} v_{i+2}^{-1}),$$

where $\Phi_4(q) = q^2 + 1$, $\Phi'_{12}(q) = q^2 + \zeta_3^2$, $\Phi''_{12}(q) = q^2 + \zeta_3$ and the indexes are taken mod 3.

Let $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and \mathfrak{p} be a prime ideal of \mathbb{Z}_K .

Definition 2.6 Let $M = \prod_{\mathcal{C}, j} v_{\mathcal{C}, j}^{a_{\mathcal{C}, j}}$ be a monomial in A such that $\gcd(a_{\mathcal{C}, j}) = 1$. We say that M is \mathfrak{p} -essential for a character $\chi \in \text{Irr}(W)$, if there exists a K -cyclotomic polynomial Ψ such that

- $\Psi(M)$ divides $s_\chi(\mathbf{v})$.
- $\Psi(1) \in \mathfrak{p}$.

We say that M is \mathfrak{p} -essential for W , if there exists a character $\chi \in \text{Irr}(W)$ such that M is \mathfrak{p} -essential for χ .

Example 2.7 The monomials $v_0^2 v_1^{-1} v_2^{-1}$, $v_1^2 v_2^{-1} v_0^{-1}$ and $v_2^2 v_0^{-1} v_1^{-1}$ are 2-essential for the irreducible character of degree 3 of G_4 .

The following proposition ([5], Prop. 3.2.6) gives a characterization of \mathfrak{p} -essential monomials, which plays an essential role in the proof of theorem 4.3.

Proposition 2.8 Let $M = \prod_{\mathcal{C}, j} v_{\mathcal{C}, j}^{a_{\mathcal{C}, j}}$ be a monomial in A such that $\gcd(a_{\mathcal{C}, j}) = 1$. We set $\mathfrak{q}_M := (M - 1)A + \mathfrak{p}A$. Then

1. The ideal \mathfrak{q}_M is a prime ideal of A .
2. M is \mathfrak{p} -essential for $\chi \in \text{Irr}(W)$ if and only if $s_\chi(\mathbf{v})/\xi_\chi \in \mathfrak{q}_M$.

3 Cyclotomic Hecke algebras

Let y be an indeterminate. We set $x := y^{|\mu(K)|}$.

Definition 3.1 A cyclotomic specialization of \mathcal{H} is a \mathbb{Z}_K -algebra morphism $\phi : \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \rightarrow \mathbb{Z}_K[y, y^{-1}]$ with the following properties:

- $\phi : v_{\mathcal{C}, j} \mapsto y^{n_{\mathcal{C}, j}}$ where $n_{\mathcal{C}, j} \in \mathbb{Z}$ for all \mathcal{C} and j .
- For all $\mathcal{C} \in \mathcal{A}/W$, if z is another indeterminate, the element of $\mathbb{Z}_K[y, y^{-1}, z]$ defined by

$$\Gamma_{\mathcal{C}}(y, z) := \prod_{j=0}^{e_{\mathcal{C}}-1} (z - \zeta_{e_{\mathcal{C}}}^j y^{n_{\mathcal{C}, j}})$$

is invariant by the action of $\text{Gal}(K(y)/K(x))$.

If ϕ is a cyclotomic specialization of \mathcal{H} , the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y, y^{-1}]$ -algebra, denoted by \mathcal{H}_ϕ , which is obtained as the specialization of the $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ -algebra \mathcal{H} via the morphism ϕ . It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t .

Example 3.2 The “spetsial” Hecke algebra $\mathcal{H}^s(W)$ is the cyclotomic algebra obtained by the specialization

$$v_{\mathcal{C},0} \mapsto y, \quad v_{\mathcal{C},j} \mapsto 1 \text{ for } 1 \leq j \leq e_{\mathcal{C}} - 1, \text{ for all } \mathcal{C} \in \mathcal{A}/W.$$

The following result is proved in [5] (remarks following Thm. 3.3.3):

Proposition 3.3 *The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.*

For $y = 1$ this algebra specializes to the group algebra KW . Thus, by “Tits’ deformation theorem”, the specialization $v_{\mathcal{C},j} \mapsto 1$ defines the following bijections

$$\begin{array}{ccccc} \text{Irr}(K(\mathbf{v})\mathcal{H}) & \leftrightarrow & \text{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow & \text{Irr}(W) \\ \chi_{\mathbf{v}} & \mapsto & \chi_\phi & \mapsto & \chi. \end{array}$$

The following result is an immediate consequence of Theorem 2.4.

Proposition 3.4 *The Schur element $s_{\chi_\phi}(y)$ associated with the irreducible character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form*

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_{\chi,\phi}} \Phi(y)^{n_{\chi,\Phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\Phi} \in \mathbb{N}$ and $C_{\chi,\phi}$ is a set of K -cyclotomic polynomials.

4 Rouquier blocks

Definition 4.1 *We call Rouquier ring of K and denote by \mathcal{R} the \mathbb{Z}_K -subalgebra of $K(y)$*

$$\mathcal{R} := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)_{n \geq 1}^{-1}]$$

Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and \mathcal{H}_ϕ the corresponding cyclotomic Hecke algebra. The *Rouquier blocks* of \mathcal{H}_ϕ are the blocks of the algebra $\mathcal{R}\mathcal{H}_\phi$.

Remark: It has been shown by Rouquier [22] that if W is a Weyl group and \mathcal{H}_ϕ is the “spetsial” cyclotomic Hecke algebra (see ex. 3.2), then its Rouquier blocks coincide with the “families of characters” defined by Lusztig.

Due to the form of the cyclotomic Schur elements, the form of the prime ideals of the Rouquier ring (see, for example, [5], Prop. 3.4.2) and an elementary result of blocks theory (see, for example, [3], Prop. 1.13), we obtain the following description of the Rouquier blocks:

Proposition 4.2 *Two characters $\chi, \psi \in \text{Irr}(W)$ are in the same Rouquier block of \mathcal{H}_ϕ if and only if there exists a finite sequence $\chi_0, \chi_1, \dots, \chi_n \in \text{Irr}(W)$ and a finite sequence $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of prime ideals of \mathbb{Z}_K such that*

- $\chi_0 = \chi$ and $\chi_n = \psi$,
- for all j ($1 \leq j \leq n$), the characters χ_{j-1} and χ_j belong to the same block of $\mathcal{R}_{\mathfrak{p}_j} \mathcal{H}_\phi$.

The above proposition implies that if we know the blocks of the algebra $\mathcal{R}_{\mathfrak{p}} \mathcal{H}_\phi$ for every prime ideal \mathfrak{p} of \mathbb{Z}_K , then we know the Rouquier blocks of \mathcal{H}_ϕ . In order to determine the former, we can use the following theorem ([7], Thm. 2.5)

Theorem 4.3 *Let $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and \mathfrak{p} be a prime ideal of \mathbb{Z}_K . Let M_1, \dots, M_k be all the \mathfrak{p} -essential monomials for W such that $\phi(M_j) = 1$ for all $j = 1, \dots, k$. Set $\mathfrak{q}_0 := \mathfrak{p}A$, $\mathfrak{q}_j := \mathfrak{p}A + (M_j - 1)A$ for $j = 1, \dots, k$ and $\mathcal{Q} := \{\mathfrak{q}_0, \mathfrak{q}_1, \dots, \mathfrak{q}_k\}$. Two irreducible characters $\chi, \psi \in \text{Irr}(W)$ are in the same block of $\mathcal{R}_{\mathfrak{p}} \mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \dots, \chi_n \in \text{Irr}(W)$ and a finite sequence $\mathfrak{q}_{j_1}, \dots, \mathfrak{q}_{j_n} \in \mathcal{Q}$ such that*

- $\chi_0 = \chi$ and $\chi_n = \psi$,
- for all i ($1 \leq i \leq n$), the characters χ_{i-1} and χ_i are in the same block of $A_{\mathfrak{q}_{j_i}} \mathcal{H}$.

Let \mathfrak{p} be a prime ideal of \mathbb{Z}_K and $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ a cyclotomic specialization. If $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ is a \mathfrak{p} -essential monomial for W , then

$$\phi(M) = 1 \Leftrightarrow \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0.$$

The hyperplane defined in $\mathbb{C}^{\sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}}$ by the relation

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0,$$

where $(t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of $\sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}$ indeterminates, is called \mathfrak{p} -essential hyperplane for W . A hyperplane in $\mathbb{C}^{\sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}}$ is called essential for W , if it is \mathfrak{p} -essential for some prime ideal \mathfrak{p} of \mathbb{Z}_K .

Example 4.4 The essential hyperplanes of G_4 are:

$$\begin{aligned} H_{0,1} : t_0 - t_1 = 0, \quad H_{0,2} : t_0 - t_2 = 0, \quad H_{1,2} : t_1 - t_2 = 0, \\ H_0 : 2t_0 - t_1 - t_2 = 0, \quad H_1 : 2t_1 - t_2 - t_0 = 0, \quad H_2 : 2t_2 - t_0 - t_1 = 0. \end{aligned}$$

Let $\phi_\emptyset : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the $n_{\mathcal{C},j}$ belong to no essential hyperplane. We call *Rouquier blocks associated with no essential hyperplane* and denote by \mathcal{B}^\emptyset the partition of $\text{Irr}(W)$ into the Rouquier blocks of $\mathcal{H}_{\phi_\emptyset}$. Now let H be an essential hyperplane for W and let $\phi_H : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the $n_{\mathcal{C},j}$ belong to the essential hyperplane H and no other. We call *Rouquier blocks associated with the essential hyperplane H* and denote by \mathcal{B}^H the partition of $\text{Irr}(W)$ into the Rouquier blocks of \mathcal{H}_{ϕ_H} . Due to theorem 4.3, the partition \mathcal{B}^H is coarser than the partition \mathcal{B}^\emptyset .

The following result is an immediate consequence of proposition 4.2 and theorem 4.3.

Proposition 4.5 *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization. If the $n_{\mathcal{C},j}$ belong to no essential hyperplane for W , then the Rouquier blocks of \mathcal{H}_ϕ coincide with the partition \mathcal{B}^\emptyset . Otherwise, let \mathcal{E} be the set of all essential hyperplanes that the $n_{\mathcal{C},j}$ belong to. Two irreducible characters $\chi, \psi \in \text{Irr}(W)$ belong to the same Rouquier block of \mathcal{H}_ϕ if and only if there exist a finite sequence $\chi_0, \chi_1, \dots, \chi_n \in \text{Irr}(W)$ and a finite sequence $H_1, \dots, H_n \in \mathcal{E}$ such that*

- $\chi_0 = \chi$ and $\chi_n = \psi$,
- for all i ($1 \leq i \leq n$), the characters χ_{i-1} and χ_i belong to the same part of \mathcal{B}^{H_i} .

Example 4.6 Let $\phi^s : v_i \mapsto y^{n_i}$ be the “spetsial” cyclotomic specialization for G_4 , i.e., $n_0 = 1$ and $n_1 = n_2 = 0$. The integers n_i belong only to the essential hyperplane $H_{1,2} : t_1 - t_2 = 0$ and therefore, the Rouquier blocks of \mathcal{H}^s coincide with the partition $\mathcal{B}^{H_{1,2}}$.

In the fourth chapter of [5], we explain how we have obtained the partitions \mathcal{B}^\emptyset and \mathcal{B}^H for every essential hyperplane H for every exceptional irreducible complex reflection group.

5 Functions a and A

Following the notations in [4], 6B, for every element $P(y) \in \mathbb{C}(y)$, we call

- *valuation of $P(y)$ at y* and denote by $\text{val}_y(P)$ the order of $P(y)$ at 0 (we have $\text{val}_y(P) < 0$ if 0 is a pole of $P(y)$ and $\text{val}_y(P) > 0$ if 0 is a zero of $P(y)$),

- degree of $P(y)$ at y and denote by $\deg_y(P)$ the negative of the valuation of $P(1/y)$.

For $\chi \in \text{Irr}(W)$, we define

$$a_{\chi_\phi} := \text{val}_y(s_{\chi_\phi}(y)) \text{ and } A_{\chi_\phi} := \deg_y(s_{\chi_\phi}(y)).$$

The proof of the following result can be found in [3], Prop. 2.9.

Proposition 5.1 *Let $\chi, \psi \in \text{Irr}(W)$. If χ_ϕ and ψ_ϕ belong to the same Rouquier block, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

In the next section, we are going to prove that the functions a and A are constant on the Rouquier blocks of the cyclotomic Hecke algebras of the exceptional irreducible complex reflection groups. In order to do that, we need to prove some results concerning the valuation and the degree of the Schur elements which hold for all complex reflection groups. First, let us introduce the symbols $(y^n)^+$ and $(y^n)^-$.

Definition 5.2 *Let $n \in \mathbb{Z}$.*

- $(y^n)^+ = \begin{cases} n, & \text{if } n > 0; \\ 0, & \text{if } n \leq 0. \end{cases}$
- $(y^n)^- = \begin{cases} n, & \text{if } n < 0; \\ 0, & \text{if } n \geq 0. \end{cases}$

Now let $\chi \in \text{Irr}(W)$. Following the notations of Theorem 2.4, the generic Schur element $s_\chi(\mathbf{v})$ associated to χ is an element of $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ of the form

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}. \quad (\dagger)$$

We fix the factorization (\dagger) for $s_\chi(\mathbf{v})$.

Proposition 5.3 *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization. Then*

- $a_{\chi_\phi} = \sum_{\mathcal{C},j} b_{\mathcal{C},j} n_{\mathcal{C},j} + \sum_{i \in I_\chi} n_{\chi,i} \deg(\Psi_{\chi,i})(\phi(M_{\chi,i}))^-.$
- $A_{\chi_\phi} = \sum_{\mathcal{C},j} b_{\mathcal{C},j} n_{\mathcal{C},j} + \sum_{i \in I_\chi} n_{\chi,i} \deg(\Psi_{\chi,i})(\phi(M_{\chi,i}))^+.$

Example 5.4 Let ϕ be a cyclotomic specialization for G_4 . Following the factorization of the generic Schur element of the character θ in example 2.5, we have that

- $a_{\theta_\phi} = 6 \cdot (\phi(v_0^2 v_1^{-1} v_2^{-1})^- + \phi(v_1^2 v_2^{-1} v_0^{-1})^- + \phi(v_2^2 v_0^{-1} v_1^{-1})^-),$
- $A_{\theta_\phi} = 6 \cdot (\phi(v_0^2 v_1^{-1} v_2^{-1})^+ + \phi(v_1^2 v_2^{-1} v_0^{-1})^+ + \phi(v_2^2 v_0^{-1} v_1^{-1})^+).$

If ϕ is the “spetsial” cyclotomic specialization, i.e., $\phi(v_0) = y$ and $\phi(v_1) = \phi(v_2) = 1$, then

$$\begin{aligned} \phi(v_0^2 v_1^{-1} v_2^{-1})^- &= 0, & \phi(v_1^2 v_2^{-1} v_0^{-1})^- &= -1, & \phi(v_2^2 v_0^{-1} v_1^{-1})^- &= -1, \\ \phi(v_0^2 v_1^{-1} v_2^{-1})^+ &= 2, & \phi(v_1^2 v_2^{-1} v_0^{-1})^+ &= 0, & \phi(v_2^2 v_0^{-1} v_1^{-1})^+ &= 0, \end{aligned}$$

Thus, we have $a_{\theta_\phi} = -12$ and $A_{\theta_\phi} = 12$.

Definition 5.5 Let $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a monomial with $\gcd(a_{\mathcal{C},j}) = 1$ and Ψ a K -cyclotomic polynomial such that $\Psi(M)$ appears in (\dagger) . The factor degree of $\Psi(M)$ for χ with respect to (\dagger) is defined as the product

$$f_{\Psi(M)}(\mathbf{t}) = \deg(\Psi) \cdot \left(\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} \right),$$

where $\mathbf{t} = (t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of $\sum_{\mathcal{C} \in A/W} e_{\mathcal{C}}$ indeterminates. If n is the greatest positive integer such that $\Psi(M)^n$ appears in (\dagger) , then n is called the coefficient of the factor degree $f_{\Psi(M)}$ and it is denoted by $\mathbf{c}(f_{\Psi(M)})$.

Then we can define an equivalence relation on the set \mathcal{F}_χ of all factor degrees for χ with respect to (\dagger) :

Definition 5.6 Two factor degrees f_1, f_2 are equivalent, if there exists a positive number $q \in \mathbb{Q}$ such that $f_1 = qf_2$. We write $f_1 \sim f_2$.

Definition 5.7 Let \mathcal{F}_χ be the set of all factor degrees for χ with respect to (\dagger) and let ϵ be a sign map for \mathcal{F}_χ , i.e., a map $\mathcal{F}_\chi \rightarrow \{-1, 1\}$. We say that ϵ is a good sign map for \mathcal{F}_χ if it satisfies the following conditions:

1. If $f_1, f_2 \in \mathcal{F}_\chi$ with $f_1 \sim f_2$, then $\epsilon(f_1) = \epsilon(f_2)$.
2. If $f_1, f_2 \in \mathcal{F}_\chi$ with $f_1 \sim -f_2$, then $\epsilon(f_1) = -\epsilon(f_2)$.

In order to obtain the main result, we need to introduce the notions of generic valuation and generic degree of the Schur element $s_\chi(\mathbf{v})$.

Definition 5.8 Let \mathcal{F}_χ be the set of all factor degrees for χ with respect to (\dagger) and let $\epsilon : \mathcal{F}_\chi \rightarrow \{-1, 1\}$ be a good sign map for \mathcal{F}_χ . Then

- the generic valuation $a_{\chi,\epsilon}(\mathbf{t})$ of $s_\chi(\mathbf{v})$ with respect to ϵ is

$$a_{\chi,\epsilon}(\mathbf{t}) := \sum_{\mathcal{C},j} b_{\mathcal{C},j} t_{\mathcal{C},j} + \sum_{\{f \in \mathcal{F}_\chi \mid \epsilon(f) = -1\}} \mathbf{c}(f) \cdot f.$$

- the generic degree $A_{\chi,\epsilon}(\mathbf{t})$ of $s_\chi(\mathbf{v})$ with respect to ϵ is

$$A_{\chi,\epsilon}(\mathbf{t}) := \sum_{\mathcal{C},j} b_{\mathcal{C},j} t_{\mathcal{C},j} + \sum_{\{f \in \mathcal{F}_\chi \mid \epsilon(f) = 1\}} \mathbf{c}(f) \cdot f.$$

The following result is a consequence of the above definitions and proposition 5.3.

Proposition 5.9 *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and $\chi, \psi \in \text{Irr}(W)$ with sets of factor degrees $\mathcal{F}_\chi, \mathcal{F}_\psi$ respectively. If $a_{\chi,\epsilon}(\mathbf{t}) = a_{\psi,\epsilon}(\mathbf{t})$ (resp. $A_{\chi,\epsilon}(\mathbf{t}) = A_{\psi,\epsilon}(\mathbf{t})$) with respect to every good sign map ϵ for $\mathcal{F}_\chi \cup \mathcal{F}_\psi$, then $a_{\chi_\phi} = a_{\psi_\phi}$ (resp. $A_{\chi_\phi} = A_{\psi_\phi}$).*

Proof: Let $\mathbf{n} := (n_{\mathcal{C},j})_{\mathcal{C},j}$. There exists a good sign map ϵ for $\mathcal{F}_\chi \cup \mathcal{F}_\psi$ such that

$$\epsilon(f) = -1 \Leftrightarrow f(\mathbf{n}) \leq 0.$$

Then, by proposition 5.3, we have that

$$a_{\chi_\phi} = a_{\chi,\epsilon}(\mathbf{n}) = a_{\psi,\epsilon}(\mathbf{n}) = a_{\psi_\phi}$$

and

$$A_{\chi_\phi} = A_{\chi,\epsilon}(\mathbf{n}) = A_{\psi,\epsilon}(\mathbf{n}) = A_{\psi_\phi}.$$

■

Corollary 5.10 *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the integers $n_{\mathcal{C},j}$ belong to the essential hyperplane $H : \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0$. Then we can assume that the set \mathbf{t} is not algebraically independent, but satisfies $\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0$. If $a_{\chi,\epsilon}(\mathbf{t}) = a_{\psi,\epsilon}(\mathbf{t})$ (resp. $A_{\chi,\epsilon}(\mathbf{t}) = A_{\psi,\epsilon}(\mathbf{t})$) with respect to every good sign map ϵ for $\mathcal{F}_\chi \cup \mathcal{F}_\psi$, then $a_{\chi_\phi} = a_{\psi_\phi}$ (resp. $A_{\chi_\phi} = A_{\psi_\phi}$).*

6 Exceptional complex reflection groups

In this section we will prove the following result

Theorem 6.1 *Let W be an exceptional irreducible complex reflection group. Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and $\chi, \psi \in \text{Irr}(W)$. If χ_ϕ and ψ_ϕ belong to the same Rouquier block, then*

$$a_{\chi_\phi} = a_{\psi_\phi} \text{ and } A_{\chi_\phi} = A_{\psi_\phi}.$$

6.1 The groups $G_4, \dots, G_{22}, G_{25}, G_{26}, G_{28}, G_{32}$

Let $W := G_m$, where $m \in \{7, 11, 19, 26, 28, 32\}$. We have created the following algorithm which verifies that the assumptions of corollary 5.10 are satisfied on the Rouquier blocks associated with each essential hyperplane. This algorithm requires the GAP package CHEVIE and the function *AllBlocks* contained in the file “RouquierBlocks.g”. A program implementing this algorithm can be found on my webpage.

Algorithm

1. We assume that there exists a function $ismultiple(g, f)$ which takes two polynomials f, g and returns
 - 1, if there exists a rational $q > 0$ such that $g = q * f$,
 - -1 , if there exists a rational $q < 0$ such that $g = q * f$,
 - 0, otherwise.
2. We define a function $FactorDegrees(H, \chi)$, where
 - H is either the list of the coefficients $a_{C,j}$ of the indeterminates in an essential hyperplane for W or the empty list in the case of “no essential hyperplane”,
 - $\chi \in \text{Irr}(W)$ is represented by its position in the list of characters of W .

In the GAP package CHEVIE, the functions *SchurModels* and *SchurData* provide us with the irreducible factors and the coefficients of the generic Schur elements of W . The function $FactorDegrees(H, \chi)$ returns a pair $[F, C]$, where F is the list of factor degrees of the Schur element of χ (a list of polynomials) and C is the term of the generic valuation (and generic degree) induced by the monomial factor N_χ .

3. We assume that there exists a function $SymmetricDifferenceWithMultiplicities(l_1, l_2)$, where l_1, l_2 are two lists, which returns a sublist l of $l_1 \cup l_2$ such that: $x \in l$ if and only if the multiplicity of x in l_1 is different than the multiplicity of x in l_2 .
4. The function $compare(a, b)$ will check the assumptions of corollary 5.10 for two irreducible characters χ, ψ . It returns “true”, if they are satisfied. In order to do that, it takes the corresponding outputs of the function $FactorDegrees$, $a := [F_\chi, C_\chi]$ and $b := [F_\psi, C_\psi]$, and sets $l := SymmetricDifferenceWithMultiplicities(F_\chi, F_\psi)$.

If l is empty, then the function returns “true”. If not, then we have to generate all good sign maps only for l , since the common factors don’t affect the result:

Step 1: We create a sublist k of l such that:

- (a) every element of l is a multiple by a non-zero rational number of an element of k ,
- (b) if $f, g \in k$ then $ismultiple(f, g) = 0$.

Step 2: We create a list a_1 as follows: For all $f \in F_\chi$, we set $p :=$ the position of the g in k such that $ismultiple(f, g) \neq 0$. If $p \neq \text{false}$, then we add to a_1 the triplet $[f, p, ismultiple(f, k[p])]$. We create a similar

list b_1 for F_ψ .

Step 3: We create all good sign maps for l which consists of creating all the lists of signs of the same length as k . Let M be such a matrix and $f \in l$. Then there exists a triplet of the form $[f, p, \text{ismultiple}(k[p], f)]$ in a_1 or b_1 . The corresponding good sign map ϵ is given by $\epsilon(f) := \text{ismultiple}(f, k[p]) \cdot M[p]$.

Step 4: We compare $a_{\chi, \epsilon}(\mathbf{t})$ with $a_{\psi, \epsilon}(\mathbf{t})$ and $A_{\chi, \epsilon}(\mathbf{t})$ with $A_{\psi, \epsilon}(\mathbf{t})$ (considering only the non-common terms) with respect to every good sign map ϵ for l , given that the condition $\sum_{c,j} ac_{c,j}tc_{c,j} = 0$ is satisfied.

5. We create a function *compareblock*(H, B), where H is a list representing one or no essential hyperplane as in *FactorDegrees* and B is a block represented as a list of integers, each of which is the position of a character in the list of characters of W . If $\text{Length}(B) = 1$, then it returns "true". If not, then it applies *FactorDegrees*(H, χ) to all the elements χ of B , creating thus the list Sch , and then returns "true" if $\text{compare}(Sch[1], Sch[j]) = \text{true}$ for all $j \in \{2, \dots, \text{Length}(B)\}$.
6. Finally, we create a function *CheckTheorem*(m) which generates the group G_m and applies *compareblock*(H, B) to every $B \in \mathcal{B}^H$, where H runs over the set $\{\emptyset, \text{essential hyperplanes for } W\}$.

The function *CheckTheorem*(m) has returned "true" for all $m \in \{7, 11, 19, 26, 28, 32\}$. Then corollary 5.10 in combination with proposition 4.5, imply that the assertion of Theorem 6.1 holds for W .

Now let $W := G_m, m \in \{4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 21, 22, 25\}$. The fact that Theorem 6.1 holds for the groups G_7, G_{11}, G_{19} and G_{26} and the use of Clifford theory for the determination of the Schur elements and the Rouquier blocks of the cyclotomic Hecke algebras associated to W (see Appendix) imply that the assertion of Theorem 6.1 holds for W .

6.2 The other exceptional groups

Let W be one of the remaining exceptional irreducible complex reflection groups: $G_{23}, G_{24}, G_{27}, G_{29}, G_{30}, G_{31}, G_{33}, G_{34}, G_{35}, G_{36}, G_{37}$. Then W is generated by reflections of order 2 whose reflecting hyperplanes belong to one single orbit under the action of W . Its generic Hecke algebra is defined over a Laurent polynomial ring in two indeterminates, v_0 and v_1 , and the only possible essential monomial is $v_0v_1^{-1}$. Therefore, its generic Schur elements can be expressed as products of K -cyclotomic polynomials in the one variable $v := v_0v_1^{-1}$, i.e., the generic Schur element $s_\chi(v)$ associated to

the irreducible character χ is an element of $\mathbb{Z}_K[v, v^{-1}]$ of the form

$$s_\chi(v) = \xi_\chi v^{b_\chi} \prod_{\Psi \in C_\chi} \Psi_\chi(v)^{n_{\chi, \Psi}},$$

where $\xi_\chi \in \mathbb{Z}_K$, $b_\chi \in \mathbb{Z}$, C_χ is a set of K -cyclotomic polynomials and $n_{\chi, \Psi} \in \mathbb{N}$. If $\phi : v \mapsto y^n$ ($n \in \mathbb{Z}$) is a cyclotomic specialization, then

- $a_{\chi_\phi} = n \cdot \text{val}_v(s_\chi(v))$.
- $A_{\chi_\phi} = n \cdot \deg_v(s_\chi(v))$.

Therefore, in order to verify theorem 6.1 for W , it suffices to check whether the degree and the valuation of the generic Schur elements remain constant on the Rouquier blocks associated with no essential hyperplane. Note that the generic Schur elements coincide with the Schur elements of the “spetsial” cyclotomic Hecke algebra and the Rouquier blocks associated with no essential hyperplane coincide with its Rouquier blocks.

We can easily create an algorithm which returns “true” if the degree and the valuation of the Schur elements of the “spetsial” cyclotomic Hecke algebra remain constant on its Rouquier blocks. A program realizing this algorithm can be found on my webpage. It requires the GAP package CHEVIE and the function *RouquierBlocks* contained in the file “RouquierBlocks.g”. Since this program has returned “true” for all $m \in \{23, 24, 27, 29, 30, 31, 33, 34, 35, 36, 37\}$, we deduce that the assertion of Theorem 6.1 holds for W .

7 Appendix

Let us assume that \mathcal{O} , A and K are defined as in section 1 and that there exists a symmetrizing form t on A .

Definition 7.1 *Let \bar{A} be a subalgebra of A free and of finite rank as an \mathcal{O} -module. We say that \bar{A} is a symmetric subalgebra of A , if it satisfies the following two conditions:*

1. \bar{A} is free (of finite rank) as an \mathcal{O} -module and the restriction $\text{Res}_{\bar{A}}^A(t)$ of the form t to \bar{A} is a symmetrizing form for \bar{A} ,
2. A is free (of finite rank) as an \bar{A} -module for the action of left multiplication by the elements of \bar{A} .

From now on, let us suppose that \bar{A} is a symmetric subalgebra of A . Moreover, let K be a finite Galois extension F such that the algebras KA and $K\bar{A}$ are both split semisimple.

Definition 7.2 *We say that a symmetric \mathcal{O} -algebra (A, t) is the twisted symmetric algebra of a finite group G over the subalgebra \bar{A} , if the following conditions are satisfied:*

- \bar{A} is a symmetric subalgebra of A ,
- There exists a family $\{A_g \mid g \in G\}$ of \mathcal{O} -submodules of A such that
 - (a) $A = \bigoplus_{g \in G} A_g$,
 - (b) $A_1 = \bar{A}$,
 - (c) $A_g A_h = A_{gh}$ for all $g, h \in G$,
 - (d) $t(A_g) = 0$ for all $g \in G, g \neq 1$,
 - (e) $A_g \cap A^\times \neq \emptyset$ for all $g \in G$ (where A^\times is the set of units of A).

Lemma 7.3 *Let $a_g \in A_g$ such that a_g is a unit in A . Then*

$$A_g = a_g \bar{A} = \bar{A} a_g.$$

Proof: Since $a_g \in A_g$, property (b) implies that $a_g^{-1} \in A_{g^{-1}}$. If $a \in A_g$, then $a_g^{-1} a \in A_1 = \bar{A}$. We have $a = a_g a_g^{-1} a \in a_g \bar{A}$ and thus $A_g \subseteq a_g \bar{A}$. Property (b) implies the inverse inclusion. In the same way, we show that $A_g = \bar{A} a_g$. ■

Sometimes the Hecke algebra of a group W' appears as a symmetric subalgebra of the Hecke algebra of another group W , which contains W' . Therefore, it would be helpful, if we could obtain the Schur elements (resp. the blocks) of the former from the Schur elements (resp. the blocks) of the latter. This is possible with the use of a generalization of some classic results, known as “Clifford theory” (see, for example, [9]), to the twisted symmetric algebras of finite groups and more precisely of finite cyclic groups.

Let W be a complex reflection group and let us denote by $\mathcal{H}(W)$ its generic Hecke algebra. Let $\mathcal{H}(W)_{\text{sp}}$ be the algebra obtained from $\mathcal{H}(W)$ by specializing some of the parameters. Let W' be another complex reflection group such that $\mathcal{H}(W)_{\text{sp}}$ is the twisted symmetric algebra of a finite **cyclic** group G over the symmetric subalgebra $\mathcal{H}(W')$. In all the cases that will be studied below, applying “Clifford theory” (cf., for example, [3], Prop. 1.42 and 1.45) gives that

1. if we denote by χ' the (irreducible) restriction to $\mathcal{H}(W')$ of an irreducible character $\chi \in \text{Irr}(\mathcal{H}(W)_{\text{sp}})$, then their Schur elements verify

$$s_\chi = |W : W'| s_{\chi'},$$

2. the blocks of the algebras $\mathcal{H}(W)_{\text{sp}}$ and $\mathcal{H}(W')$ coincide.

The groups G_4, G_5, G_6, G_7

The following table gives the specializations of the parameters of the generic Hecke algebra $\mathcal{H}(G_7)$, $(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2)$, which give the generic Hecke algebras of the groups G_4, G_5 and G_6 ([17], Table 4.6).

Group	Index	S	T	U
G_7	1	x_0, x_1	y_0, y_1, y_2	z_0, z_1, z_2
G_5	2	$1, -1$	y_0, y_1, y_2	z_0, z_1, z_2
G_6	3	x_0, x_1	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2
G_4	6	$1, -1$	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2

Specializations of the parameters for $\mathcal{H}(G_7)$

The groups $G_8, G_9, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}$

The following table gives the specializations of the parameters of the generic Hecke algebra $\mathcal{H}(G_{11})$, $(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3)$, which give the generic Hecke algebras of the groups G_8, \dots, G_{15} ([17], Table 4.9).

Group	Index	S	T	U
G_{11}	1	x_0, x_1	y_0, y_1, y_2	z_0, z_1, z_2, z_3
G_{10}	2	$1, -1$	y_0, y_1, y_2	z_1, z_1, z_2, z_3
G_{15}	2	x_0, x_1	y_0, y_1, y_2	$\sqrt{u_0}, \sqrt{u_1}, -\sqrt{u_0}, -\sqrt{u_1}$
G_9	3	x_0, x_1	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3
G_{14}	4	x_0, x_1	y_0, y_1, y_2	$1, i, -1, -i$
G_8	6	$1, -1$	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3
G_{13}	6	x_0, x_1	$1, \zeta_3, \zeta_3^2$	$\sqrt{u_0}, \sqrt{u_1}, -\sqrt{u_0}, -\sqrt{u_1}$
G_{12}	12	x_0, x_1	$1, \zeta_3, \zeta_3^2$	$1, i, -1, -i$

Specializations of the parameters for $\mathcal{H}(G_{11})$

The groups $G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}$

The following table gives the specializations of the parameters of the generic Hecke algebra $\mathcal{H}(G_{19})$, $(x_0, x_1; y_0, y_1, y_2; z_0, z_1, z_2, z_3, z_4)$, which give the generic Hecke algebras of the groups G_{16}, \dots, G_{22} ([17], Table 4.12).

Group	Index	S	T	U
G_{19}	1	x_0, x_1	y_0, y_1, y_2	z_0, z_1, z_2, z_3, z_4
G_{18}	2	$1, -1$	y_0, y_1, y_2	z_0, z_1, z_2, z_3, z_4
G_{17}	3	x_0, x_1	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3, z_4
G_{21}	5	x_0, x_1	y_0, y_1, y_2	$1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$
G_{16}	6	$1, -1$	$1, \zeta_3, \zeta_3^2$	z_0, z_1, z_2, z_3, z_4
G_{20}	10	$1, -1$	y_0, y_1, y_2	$1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$
G_{22}	15	x_0, x_1	$1, \zeta_3, \zeta_3^2$	$1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$

Specializations of the parameters for $\mathcal{H}(G_{19})$

The groups G_{25} , G_{26}

The following table gives the specialization of the parameters of the generic Hecke algebra $\mathcal{H}(G_{26})$, $(x_0, x_1; y_0, y_1, y_2)$, which give the generic Hecke algebra of the group G_{25} ([19], Theorem 6.3).

Group	Index	S	T
G_{26}	1	x_0, x_1	y_0, y_1, y_2
G_{25}	2	$1, -1$	y_0, y_1, y_2

Specialization of the parameters for $\mathcal{H}(G_{26})$

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